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Swirling flow boundary layers

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The modified Oseen method is extended to provide a description of the boundary layers which accompany certain swirling flows over a rigid boundary in a rotating container. By comparison with known results it is shown that a refined procedure has errors of the order of 1 % when the inviscid flow is a rigid body rotation; it is anticipated that, for the more interesting flows, the error is of the order of 30 %.

1. Introduction

In order to pursue successfully the study of the dynamics of intense storms one needs a convenient and reasonably accurate scheme for the calculation of the time-dependent flow in the boundary layer which lies between an essentially inviscid, intensely swirling flow and a rotating rigid boundary. Numerical procedures can be (and have been) used with some success but it is hard to determine whether some of the features of the description of the flow so obtained are real or whether they are artifacts of the computational technique. Here, we evolve Oseen-like treatment of such flow fields; the details of the technique differ markedly from those which have been used on other flow configurations. It provides very useful results which seem to be very accurate.

2. Steady flow

Suppose that the steady horizontal velocity components of the inviscid flow in which we are interested are given by $U = 0$, $V = V(r)$, in $r_1 < r < r_0$ with $V(r_0) = 0$ and with $r_1/r_0 \ll 1$. Suppose also that the boundary layer which lies between this flow and the rigid boundary at $z = 0$ is steady and that the vertical component of velocity in the inviscid flow is that which is implied by the dynamics of the boundary layer. That is, if $w(x,z)$ is the vertical velocity in the boundary layer and $W(r)$ is that in the inviscid flow, then

$$W(r) = w(r,\infty).$$

The equations implying the conservation of momentum and of mass for an axially symmetric boundary-layer flow can be written in the form

$$uu_r + wu_z - (v^2/r) - 2\Omega v + \rho^{-1}p_r = \nu u_{zz}, \tag{2.1}$$

$$u(rv)_r + w(rv)_z + 2\Omega(ru) = \nu(rv)_{zz}, \tag{2.2}$$

$$rw_z + (ru)_r = 0, \tag{2.3}$$

where $\Omega\hat{z}$ denotes the angular velocity of the container, ν is the kinematic viscosity of the fluid; (u, v, w) is the velocity and r, z are the co-ordinates. Horizontal diffusion of momentum is ignored. The pressure, aside from the hydrostatic effects of gravity, is given by

$$\rho^{-1}p_r = V^2/r + 2\Omega V.$$

The arguments which underlie the scheme to be used are these: equations (2.1), (2.2) and (2.3) imply a balance between the vertical diffusion of angular momentum, the contribution of the Coriolis acceleration to the angular momentum flux, and the convection of angular momentum. It is known (Lewis & Carrier 1949) that in such balances one can frequently replace the details of the description of the convective mechanism by a simpler description which has the same over-all effect. In particular, one can frequently do this with a convective term which is linear in the velocity components. The advantages which accrue when a linear approximation can be used are self-evident.

In (2.1), the radial momentum balance is implied. There, convection, centripetal acceleration, Coriolis acceleration, diffusion, and the known pressure gradient enter the balance. The convection mechanism and the centripetal acceleration provide the non-linear contributions and it is only natural to try to extend the foregoing idea by replacing $uu_r + wu_z - v^2/r$ by an appropriate linear approximation.

Unfortunately, a universal replacement recipe does not serve our purposes. There may be one, for all I know, but it is not conveniently found; neither would it be particularly convenient to use. Accordingly, we will adopt the idea in different ways for different illustrative examples. The first example is chosen because the results are so simple and because they can be checked against very accurate knowledge which is already documented. Thus, it provides a test (but not a very severe one) of the method and it is interesting in its own right. The second example involves the configuration we need to understand in connexion with the severe storm problem and it provides insight for the large family of flow configurations in which $\partial V/\partial r < 0$.

3. Solid body rotation

It is our purpose in this section to find and to rationalize the simplest approximation for the convective process which gives an accurate description of the boundary layer under the flow,

$$U(r) = 0, \quad V(r) = \mu r, \quad \text{in} \quad 0 < r < \infty.$$

For this flow (and for many others) it is easy to anticipate that

$$v(r,z) = V(r)\, F(r,z),$$

where F is of order unity and $F(r,\infty) = 1$. Alternatively, one could also anticipate that

$$u(r,z) = V(r) \,.\, G(r,z),$$

where G is of order unity, but one would bet less enthusiastically that in an Oseen-like process, G could be approximated by unity for all r and z than one would that F could be replaced by unity. Accordingly, in (2.2), we replace $u(rv)_r + w(rv)_z$ by $u(rV)_r$. Ordinarily, when one uses such a scheme, this term is replaced by $C(r)(rv)_r$ where $C(r)$ is chosen *a postieri* using some gross conservation criterion. Here, however, we are guided by (*a*) the fact that the ordinary differential equation to which this leads is much easier to deal with than the partial differential equation to which the conventional procedure leads; (*b*) the hope that the boundary layer is primarily locally controlled as in the linear situation; (*c*) the fact that it is easier to guess a good approximation to the average size (and influence) of $(rv)_r$ than of u; and (*d*) the fact that our objective is to find the *simplest* possible scheme for the calculation.

Equation (2.1) can be rewritten in the form

$$uu_r + wu_z + \frac{(V+v)(V-v)}{r} + 2\Omega(V-v) = \nu u_{zz}.$$

Here it is convincingly plausible that the centripetal acceleration is much more important than convection so, aided again by the foregoing arguments (*a*), (*b*), (*c*), (*d*) we write

$$\nu u_{zz} = -2(\Omega + V/r)(v - V). \tag{3.1}$$

Thus, (3.1) and our replacement of (2.2), i.e.

$$\nu[r(v-V)]_{zz} = \left[2\Omega + \frac{1}{r}(rV)_r\right] ru, \tag{3.2}$$

are the equations to be solved for u and v. Thus far, no argument has depended on the choice $V = \mu r$. Using that choice now, we have

$$\tfrac{1}{2}\nu[\phi + i\psi]_{zz} = i(\Omega+\mu)[\phi + i\psi], \tag{3.3}$$

where $\phi = ru$, and $\psi = r(v - V)$.

Thus

$$\phi + i\psi = -i\mu r^2 \exp\left[-\left(\frac{2i(\Omega+\mu)}{\nu}\right)^{\frac{1}{2}} z\right]. \tag{3.4}$$

Equation (2.3) implies that

$$\begin{aligned} w(r,\infty) &= -\int_0^\infty \frac{1}{r}\phi_r\,dz \\ &= -\frac{1}{r}\frac{\partial}{\partial r} R_e \int_0^\infty (\phi + i\psi)\,dz \\ &= \mu\left(\frac{\nu}{\Omega+\mu}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

One can compare this result with that found by Rogers & Lance (1960), noting that $\mu + \Omega$ is their ω. The inaccuracy at large swirl speeds is of the order of 30 %.

The refinement which greatly improves this result stems from the hypothesis that we have not really allowed the wv_z convective contributions to be felt. Accordingly, we now replace (2.1) and (2.2) by

$$\nu(\phi + i\psi)_{zz} - \alpha(\phi + i\psi)_z = 2i(\Omega+\mu)(\phi + i\psi), \tag{3.6}$$

where α represents some appropriate average of w.

The quantity, $\phi + i\psi$, now has the form

$$\phi + i\psi = -i\mu r^2 e^{-kz},$$

where

$$k = -\frac{\alpha}{2\nu} + \left[\frac{\alpha^2}{4\nu^2} + \frac{2i(\Omega+\mu)}{\nu}\right]^{\frac{1}{2}}.$$

Thus

$$w(r,\infty) = \frac{\mu}{\Omega+\mu}\left[\left\{\frac{1}{2}\left(\left[\frac{\alpha^4}{16} + 4(\Omega+\mu)^2\nu^2\right]^{\frac{1}{2}} + \tfrac{1}{4}\alpha^2\right)\right\}^{\frac{1}{2}} + \tfrac{1}{2}\alpha\right]. \tag{3.7}$$

Since α was to be an appropriate average of w, (3.7) can be regarded as an equation for $w(r,\infty)$ and we take†

$$\alpha = \tfrac{1}{2}w(r,\infty).$$

The values given by (3.7) with $\alpha = \frac{1}{2}w_\infty$ are very good indeed (see table 1) and so is the location of the zeros of ϕ and ψ. (See Lance & Rogers 1960.)

Thus, this analysis certainly suffices for the rigid body rotation problem but the test is not a critical one. The exact solution for $(\phi + i\psi)/r^2$ is independent of r, w is a constant (for a given μ) and it is clear that more subtle difficulties might arise in more intricate problems.

$\Omega/(\Omega+\mu)$	$w(r,\infty)$, this study	$w(r,\infty)$, Lance & Rogers
1	0	0
0·81	0·2	0·2
0·63	0·4	0·39
0·33	0·8	0·83
0	1·4	1·37

TABLE 1

4. The flow with $rV = A(1-r^2/r_0^2)$

The procedure which led to (3.1) and (3.2) can be applied without change to the problem in which

$$\Psi \equiv rV = A(1-(r^2/r_0^2)), \quad U = 0, \quad \text{in} \quad r_1 < r < r_0$$

with $u(r_0,z) = v(r_0,z) = 0$. Instead of leading to (3.3), it leads to the differential equations

$$\nu\phi_{zz} = -2(\Omega + \Psi/r^2)\psi, \tag{4.1}$$

$$\nu\psi_{zz} = 2(\Omega + (1/2r)\Psi_r)\phi. \tag{4.2}$$

These imply that

$$\phi + iN\psi = -iN\Psi \exp[-(2i/\nu)^{\frac{1}{2}}\lambda z], \tag{4.3}$$

where

$$N^2 = (\Omega + \Psi/r^2)/(\Omega + \Psi_r/2r) \tag{4.4}$$

and

$$\lambda^4 = (\Omega + \Psi/r^2)(\Omega + \Psi_r/2r). \tag{4.5}$$

It then follows that

$$w(r,\infty) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{N\Psi\nu^{\frac{1}{2}}}{\lambda}\right). \tag{4.6}$$

† To see that one *knows* $\frac{1}{2}w_\infty$ is a good average, try the simple example $w_{zz} - ww_z = 0$, with $w(0) = 0$, $w(\infty) = -1$.

That the result is completely unreasonable for small r is easy to discover. According to (4.3), with $z_1 = \pi(\nu/2\lambda)^{\frac{1}{2}}$,

$$\phi(r, z_1) = -N\Psi e^{-\frac{1}{2}\pi}$$

and this gets much larger than Ψ as r gets small.

One might imagine that this could again be attributed to the omission of the $w\phi_z$ and $w\psi_z$ terms but such is not the case; if one carries out the refinement of §3 for this configuration, the result is still unreasonable (i.e. the analysis still predicts $\phi/\Psi \gg 1$). The fault lies in our convective approximation in the radial momentum equation. When the equation is written in terms of ϕ and ψ it has the form

$$\frac{1}{r}\phi\phi_r + w\phi_z - 2\Omega\psi - \frac{(2\Psi+\psi)\psi}{r^2} - \frac{\phi^2}{r^2} = \nu\phi_{zz}, \tag{4.7}$$

where ψ still denotes $r(v-V)$.

We can expect ϕ to change with r in proportion to Ψ so that, for small r,

$$\frac{1}{r}\phi\phi_r = O\left(\frac{A^2}{r_0^2}\right),$$

but

$$\frac{\phi^2}{r^2} = O\left(\frac{A^2}{r^2}\right).$$

The results of (4.3) allow the omitted ϕ^2/r^2 term to be bigger than each of the terms we retained in that equation; thus, if the results are to be useful, it becomes clear that, in an improved procedure, we must retain a suitable approximation for ϕ^2/r^2 in the linearized form of (4.7). It also seems clear that, as r gets small, $(2\Psi+\psi)\psi + \phi^2/r^2$ must remain reasonably small except near $z = 0$ where it cannot do so because of the boundary condition. Physically, this suggests that, at small r/r_0, there will be a frictionally controlled flow very close to $z = 0$ but, in the outer part of the boundary layer, the radial momentum balance will be one in which friction plays no role but in which

$$(2\Psi+\psi)\psi + \phi^2 \simeq 0.$$

That is, the outer part of the boundary layer will be governed by a balance among centripetal acceleration, pressure gradient and radial convection of momentum.

To reflect this in the mathematics, we write instead of (4.7)

$$\nu\phi_{zz} = -2(\Omega + D\Psi/r^2) + (C\Psi/r^2)\phi. \tag{4.8}$$

We will choose the positive constants, C, D when we have found ϕ and can require internal consistency of the result.

We must now solve (4.2) and (4.8). We write

$$z = \phi + \beta\psi, \tag{4.9}$$

and combine (4.2) and (4.8) in the form

$$\nu[\phi_{zz} + \beta\psi_{zz}] = -2(\Omega + D\Psi/r^2)\psi + [C\Psi/r^2 + 2\beta(\Omega + (1/2r)\Psi_r)]\phi. \tag{4.10}$$

We now require that

$$\beta = \frac{-2(\Omega + D\Psi/r^2)}{2\beta(\Omega + (1/2r)\Psi_r) + C\Psi/r^2},$$

i.e.
$$\beta = \frac{-C\Psi/r^2 \pm [C^2\Psi^2/r^4 - 16(\Omega + (1/2r)\Psi_r)(\Omega + D\Psi/r^2)]^{\frac{1}{2}}}{4(\Omega + (1/2r)\Psi_r)}, \tag{4.11}$$

so that
$$\nu(\phi_{zz} + \beta\psi_{zz}) = a^2(\phi + \beta\psi)$$

and
$$a^2 = [C\Psi/r^2 + 2\beta(\Omega + (1/2r)\Psi_r)] = \frac{-2(\Omega + D\Psi/r^2)}{\beta}. \tag{4.12}$$

Two extremes are of immediate interest. When Ψ/r^2 is small enough, $(1 - r^2/r_0^2 \ll 1)$, both values of β are nearly imaginary. In fact, one is the conjugate of the other and the same pair of functions ϕ, ψ is obtained which ever value of β is used. This solution corresponds closely to the linear Ekman layer and to the boundary layers of §3.

At the other extreme, when Ψ/r^2 is large enough (small r/r_0), there are two real roots, β, one much larger than the other. In fact

$$\beta_1 \sim -(C\tfrac{1}{2}\Psi)(r^2\Omega + \tfrac{1}{2}r\Psi_r) \tag{4.13}$$

and
$$\beta_2 \sim -2D/C, \tag{4.14}$$

so that (ignoring Ω compared to Ψ/r^2)

$$a_1 \sim -[4(\Omega + (1/2r)\Psi_r)\, D/C]^{\frac{1}{2}}, \tag{4.15}$$

$$a_2 \sim -(C\Psi/r)^{\frac{1}{2}}. \tag{4.16}$$

Since
$$\phi + \beta_1\psi = A_1 \exp[a_1 z/\nu^{\frac{1}{2}}],$$
$$\phi + \beta_2\psi = A_2 \exp[a_2 z/\nu^{\frac{1}{2}}],$$

where A_1 and A_2 must be determined, we invoke the fact that $\phi(r, 0) = 0$ to obtain

$$A_1/\beta_1 = A_2/\beta_2$$

and we use
$$\psi(r, 0) = -\Psi(r),$$

to obtain
$$(A_1 - A_2)/(\beta_1 - \beta_2) = -\Psi,$$

so that
$$\phi = \frac{\beta_1\beta_2}{\beta_1 - \beta_2}\Psi(\exp[a_1 z/\nu^{\frac{1}{2}}] - \exp[a_2 z/\nu^{\frac{1}{2}}]), \tag{4.17}$$

and
$$\psi = -\Psi(\beta_1 \exp[a_1 z/\nu^{\frac{1}{2}}] - \beta_2 \exp[a_2 z/\nu^{\frac{1}{2}}])/(\beta_1 - \beta_2). \tag{4.18}$$

Note now that, since $2D\Psi\psi$ was used to approximate $(2\Psi + \psi)\psi$ we can now require that, in the inviscid region of the boundary layer,

$$\int_0^\infty 2D\Psi\psi\, dz = \int_0^\infty (2\Psi + \psi)\psi\, dz. \tag{4.19}$$

In the inviscid region (which is most of the z domain for small r)

$$\psi \simeq -\Psi \exp[a_1 z/\nu^{\frac{1}{2}}].$$

Thus,
$$2(1 - D)\Psi^2/a_1 = \Psi^2/2a_1, \tag{4.20}$$

i.e.
$$D = \tfrac{3}{4}. \tag{4.21}$$
It then follows that, in the inviscid region
$$u^2+v^2 \simeq V^2 \tag{4.22}$$
is a better approximation to ϕ than the description given by (4.17). However, we do not really care about the details of ϕ except in their effect on $w(r,\infty)$. Accordingly, we choose C so that the inviscid contribution to w, as given by
$$w(r,\infty) = -\frac{1}{r}\frac{\partial}{\partial r}\int_0^\infty \phi'\,dz,$$
is the same whether ϕ' is the inviscid part of ϕ as given by (4.17) or that implied by (4.22).

That is,
$$\int \beta_2 \Psi \exp[a_1 z/\nu^{\frac{1}{2}}]\,dz = \int [(2\Psi+\psi)(-\psi)]^{\frac{1}{2}}\,dz. \tag{4.23}$$
Hence
$$\frac{\Psi\beta_2}{-a_1} = \int \Psi\,[(2-\exp[a_1 z/\nu^{\frac{1}{2}}])\exp[a_1 z/\nu^{\frac{1}{2}}]]^{\frac{1}{2}}\,dz = (\tfrac{1}{2}\pi+1)\Psi/a_1$$
and, since $\beta_2 \simeq -(2D/C)$
$$C = 3/(\pi+2) \simeq \tfrac{3}{5}.$$
Thus, with $D = \frac{3}{4}$, $C = \frac{3}{5}$, and
$$w(r,\infty) = -r^{-1}\int_0^\infty \phi_r(r,z)\,dz,$$
(4.17) and (4.18) describe a flow whose *important* features are those of the real flow. In particular, they describe a flow in which fluid which is affected by friction at large r gradually emerges into a region in which the control is inertial but where the dynamics are still very important in the determination of $w(r,\infty)$. For this flow, i.e. for $\psi = A(1-r^2/r_0^2)$, one can readily infer that $w(r,\infty)$ varies very slowly in r and that, when $A = \Omega r_0^2/20$ (the geophysically interesting case), the downdraft from the outer flow in $r_0^2 > r^2 > r_0^2/25$ is well approximated by
$$w(r,\infty) \simeq \frac{-A}{r_0^2}(\nu/\Omega)^{\frac{1}{2}}.$$
Alternatively, for very small r,
$$w(r,\infty) \simeq -2(5/4)^{\frac{1}{2}}\frac{A}{r_0^2}(\nu/\Omega)^{\frac{1}{2}}.$$
Hence, even under the intense part of the swirl, $w(r,\infty)$ is only twice as large as it is in the more slowly moving air.†

† *Note added in proof.* There is an algebraic error in this calculation of $w(r,\infty)$ for small r which was noticed by J. McWilliams. The correct result is $w(r,\infty) \simeq r^{-1}(\nu\Psi/C)^{\frac{1}{2}}$, which implies that the vertical convection of momentum must play an even more important role for small r than it did in the problem of §3. McWilliams will publish in due course an account of this refinement of the analysis, but we anticipate his results here only to the extent of assuring the reader that it restores the general conclusion reached in this paper. That is, the downdraft for this phenomenon in $r_0/50 < r < r_0$ is dominated by the linear result $w(r,\infty) \simeq -Ar_0^{-2}(\nu/\Omega)^{\frac{1}{2}}$, and the use of this fact and the related results of §§5, 6 in the succeeding hurricane paper is justified.

In the absence of accurate information concerning the validity associated with the choice of an eddy viscosity, ν, or accurate estimates of its value, further refinement of the foregoing model and the understanding it provides would be hard to justify for phenomena involving turbulent flow.

5. A time-dependent flow: linear analysis

Let the inviscid flow in $R(t) < r < r_0$ be that for which

$$\Psi = \Omega[R_0^2 - R^2(t)]\frac{r_0^2 - r^2}{r_0^2 - R^2(t)}, \tag{5.1}$$

where $$\Psi = rV(r,t).$$

The radial velocity, $U(r, T)$, which accompanies this flow, in the inviscid region, must be such that the angular momentum of each particle is conserved and such that $U(R(r),t) = \dot{R}$ and $U(r_0, t) = 0$. That radial velocity is given by

$$\Phi = R(t)\,\dot{R}(t)\frac{r_0^2 - r^2}{r_0^2 - R^2(t)}, \tag{5.2}$$

where $$\Phi(r,t) = rU(r,t).$$

The low Rossby number ($\Phi^2 + \Psi^2 \ll \Omega^2 R^4$) equations for the viscous layer are

$$\nu\phi_{zz} - \phi_t = -2\Omega\psi,$$

$$\nu\psi_{zz} - \psi_t = 2\Omega\phi,$$

where $$\phi = r[u(r,z,t) - U(r,t)], \quad \psi = r[v(r,z,t) - V(r,t)],$$

i.e. $$\nu(\phi + i\psi)_{zz} - (\phi + i\psi)_t = 2\Omega i(\phi + i\psi). \tag{5.3}$$

One can use transform methods to find $\phi + i\psi$ with

$$\phi + i\psi \to -(\Phi + i\Psi) \quad \text{on} \quad z = 0$$

and $$\phi + i\psi \to 0 \quad \text{as} \quad z \to \infty,$$

where $R(t)$is any reasonable function of t. Our purposes are served, however, by studying the case where $R_0^2 - R^2 = R_0^2 e^{\alpha t}$ and where $r_0^2 - R^2$ is so close to r_0^2 that its time variation can be ignored. Under these conditions

$$\Psi \simeq \Omega R_0^2 e^{\alpha t}(1 - r^2/r_0^2) = \Psi_0 e^{\alpha t}, \tag{5.4}$$

$$\Phi \simeq -\tfrac{1}{2}\alpha R_0^2 e^{\alpha t}(1 - r^2/r_0^2) = \Phi_0 e^{\alpha t}. \tag{5.5}$$

For this case $$\phi + i\psi = \chi e^{\alpha t}, \tag{5.6}$$

and (5.3) becomes $$\nu\chi_{zz} = (\alpha + 2\Omega i)\,\chi, \tag{5.7}$$

so that $$\chi = -(\Phi_0 + i\Psi_0)\exp\left[-z\{(\alpha + 2\Omega i)/\nu\}^{\frac{1}{2}}\right]. \tag{5.8}$$

The vertical velocity can be written as the sum of two items. One is associated with the inviscid flow and is given by

$$w_i(r,z,t) = -\frac{1}{r}\frac{\partial}{\partial r}\int_0^z \Phi(r,t)\,dz = -\alpha\frac{R_0^2}{r_0^2}ze^{\alpha t}. \tag{5.9}$$

The other, $w(r,z,t)$, is associated with the boundary-layer flow, as given by ϕ and ψ. This contribution is the one which is related to the amount of fluid which (in the boundary layer) flows radially through the moving cylindrical surface, $r = R(t)$.

Thus, (with $w = W(r,z)\,e^{\alpha t}$)

$$W(r,z) = -\frac{1}{r}\frac{\partial}{\partial r}\int_0^z \phi\, dz$$

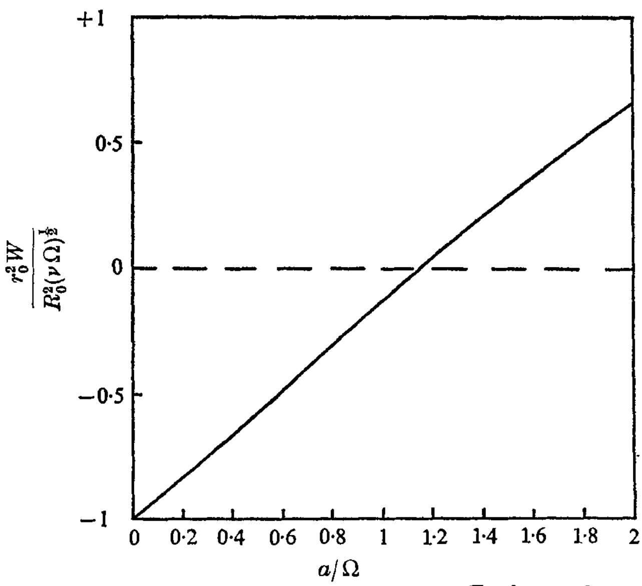

FIGURE 1. W versus α for $\Psi = \Omega R_0^2 e^{\alpha t}\left[\frac{r_0^2 - r^2}{r_0^2 - R^2(t)}\right]$.

and, in particular

$$\begin{aligned} W(r,\infty) &= -\frac{1}{r}\frac{\partial}{\partial r}\mathrm{Re}\int_0^\infty \chi\, dz \\ &= \mathrm{Re}\,\frac{1}{r}\frac{\partial}{\partial r}(\Phi_0 + i\Psi_0)\,[\nu/(\alpha + 2\Omega i)]^{\frac{1}{2}} \\ &= \frac{R_0^2}{r_0^2}\mathrm{Re}\,\frac{(\alpha - 2\Omega i)\,\nu^{\frac{1}{2}}}{[\alpha + 2\Omega i]^{\frac{1}{2}}} \\ &= \frac{R_0^2\,\nu^{\frac{1}{2}}}{r_0^2\,\rho}\,[\alpha\{\tfrac{1}{2}(\rho+\alpha)\}^{\frac{1}{2}} - 2\Omega\{\tfrac{1}{2}(\rho-\alpha)\}^{\frac{1}{2}}], \end{aligned} \qquad (5.10)$$

where $\rho = \{\alpha^2 + 4\Omega^2\}^{\frac{1}{2}}$.

The foregoing result, in which W is less negative with $\alpha > 0$ than it is with $\alpha = 0$, is a consequence of two effects (see figure 1). One effect is associated with the delayed growth of the boundary layer but an equally important effect is associated with the fact that $W(r,\infty)$ depends on the amount of fluid which the boundary layer ejects to (or receives from) the region in $r < R(t)$. When $\dot{R}$ is negative, the profile of u as a function of z looks like the sketches of figure 2.

When U is large enough compared to V, the amount of ejected fluid may actually be negative as depicted by the curve labelled u_2 but, when $|\dot{R}|$ is small enough compared to V, there will be a positive ejection rate. Thus, the second effect (a diminution of W) is merely associated with the frictional diminution of the inviscid radial velocity component.

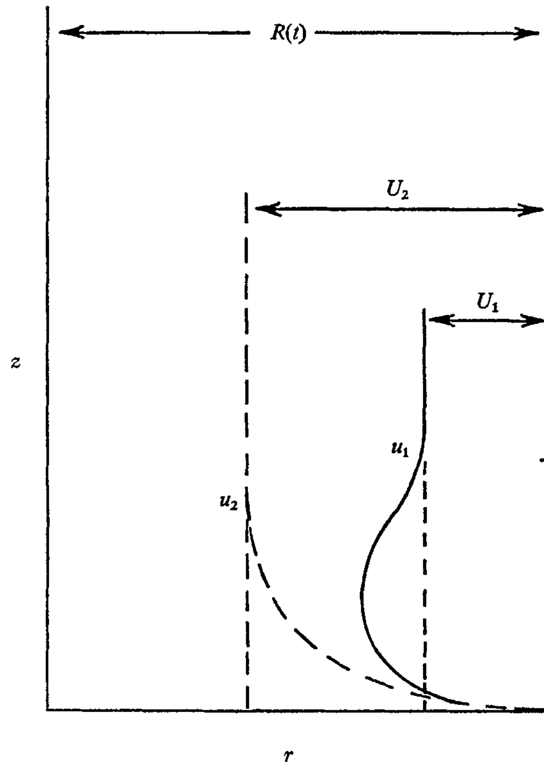

FIGURE 2. Boundary-layer velocity profiles u_1, u_2, at $r = R(t)$ for two different values of U.

In figure 1 we plot $W(r, \infty)$ (which actually does not vary with r) versus α, the parameter which characterized the relative size of Φ and Ψ. Note that, for $\alpha > 7/6\Omega$, $W(r, \infty)$ is positive. Thus, any phenomenon which involves the swirling flow studied in this section and which requires that there be a flux of fluid radially inward across the surface at $r = R(t)$ must have a time scale whose order of magnitude cannot be shorter than Ω^{-1}.

6. Intense time-dependent flows

When $\Phi^2 + \Psi^2$ is not small compared to $\Omega^2 R^4$, the foregoing linear analysis may not suffice. Once again, however, we can use the ideas of § 4 in connexion with the time-dependent conservation equations to obtain

$$\nu\phi_{zz} - \phi_t = -2(\Omega + D\Psi/r^2)\psi + (C\Psi/r^2)\phi \tag{6.1}$$

and

$$\nu\psi_{zz} - \psi_t + 2(\Omega + (1/2r)\Psi_r)\phi. \tag{6.2}$$

Furthermore, we can try again to find a combination, $\chi = \phi + \beta\psi$, for which (6.1) and (6.2) collapse into the form

$$\nu\chi_{zz} - \chi_t = a^2\chi. \tag{6.3}$$

Equation (6.3) is consistent with (6.1) and (6.2) only if

$$a^2 = 2\beta(\Omega + (1/2r)\Psi_r) + C(\Psi/r^2), \tag{6.4}$$

and

$$-a^2\beta = 2(\Omega + D\Psi/r^2) + \beta t, \tag{6.5}$$

so that

$$\beta_t + C(\Psi/r^2) + 2\beta^2(\Omega + (1/r)\Psi_r) = -2(\Omega + D\Psi/r^2). \tag{6.6}$$

When Φ is not larger than Ψ anywhere in the domain, one should retain quite reasonable accuracy by choosing $C = \frac{3}{5}$ and $D = \frac{3}{4}$ as in the steady case. Then, depending on the time dependence of Ψ, one can solve (6.6) for β by, at worst, a simple forward-in-time numerical integration scheme and one can then use (6.5) to find a^2.

With a^2 known, and for values of r at which β is complex, one can solve (6.3) by writing

$$\chi = y(z,r,t)\exp\left[-\int_0^t a^2\,dt\right] = y\,e^{-\tau(t)}, \tag{6.7}$$

so that

$$\nu y_{zz} - y_t = 0. \tag{6.8}$$

Since y is known at $z = 0$ in terms of Φ, Ψ, and β, we can then use the Laplace transform technique (with

$$\bar{y} = \int_0^\infty e^{-st}y\,dt$$

and with $y(0,r,t) = 0$) to obtain

$$\bar{y}(z,r,s) = \bar{y}(0,r,s)\exp\left[-a(s/\nu)^{\frac{1}{2}}\right]. \tag{6.9}$$

$\overline{W}(r,\infty,s)$ is related to the integral of $\bar{y}$ over z in a relatively simple way and one can obtain a convolution integral for $W(r,\infty,t)$. All of this is very simple in principle but the details are likely to be a mess. For values of r at which β becomes real with increasing t, both branches of β must be identified (calculated) and χ will be a linear combination of the appropriate solution of (6.3) with $\beta = \beta_1$ and of (6.3) with $\beta = \beta_2$. For such r and t the procedure will be even messier than that for complex β. Thus, in the absence of any compelling reason to choose a different illustration, we work out here only the details of the particular problem which led us into this study. It is the flow described in §5 with $R_0 \leqslant r_0/5$, $R_1/10 \leqslant R(t) \leqslant R_0$, $\alpha = O(\Omega)$, and we confine our attention to the region in $r \geqslant R_0$.

In the steady state version of this problem, C plays no significant role in $r > R_0$ and, in particular, β is complex. More important, the coefficients in (6.1) and (6.2) change by so little over the entire domain (in r and t) that the linear analysis is clearly much more accurate than is consistent with our lack of knowledge of the eddy viscosity ν which must be adopted in any large scale problem. Thus, the analysis of this problem is already contained in §5 and no further details are needed.

Note that the efflux of fluid across $r = R(t)$ is also accurately given by this analysis even though the details of the flow have not been studied in $R(t) < r < R_0$. This follows merely from the fact that the efflux, E, is given by

$$-E = \pi\int_R^{r_0} W_\infty r\,dr$$

and that the region $R_0 < r < r_0$ accounts for more than 96% of the region $R(t) < r < r_0$ from which fluid is being drained.

There is a delightful anomaly in this result. Despite the fact that, in part of the domain, $R_0 < r < r_0$, the Rossby number becomes as large as 10^2, the zero Rossby number theory gives a completely satisfactory evaluation of the amount of fluid transported radially by the boundary layer.

In problems where more meticulous detail is required, the foregoing methods will provide, at the very least, an excellent guide-line for the choice of functions in terms of which ϕ and ψ can be described with the help of a Galerkin method or some other computational procedure. Without such a guide-line, the functions chosen may provide (and already have sometimes provided) descriptions of somewhat limited validity and utility.

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